

EXPLICIT SOLUTIONS TO A VECTOR TIME SERIES MODEL AND ITS INDUCED MODEL FOR BUSINESS CYCLES

XIONGZHI CHEN

ABSTRACT. This article gives the explicit solution to a general vector time series model that describes interacting, heterogeneous agents that operate under uncertainties but according to Keynesian principles, from which a model for business cycle is induced by a weighted average of the growth rates of the agents in the model. The explicit solution enables a direct simulation of the time series defined by the model and better understanding of the joint behavior of the growth rates. In addition, the induced model for business cycles and its solutions are explicitly given and analyzed. The explicit solutions provide a better understanding of the mathematics of these models and the econometric properties they try to incorporate.

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1. INTRODUCTION

The study and modeling of business cycles have significant importance in economic theory and practice; see, e.g., [Burns and Mitchell \(1946\)](#), [Lucas \(1977\)](#), [Kydland and Prescott](#)

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(1977) and Kydland and Prescott (1982). Models of business cycles have various forms as surveyed in Ahking and Miller (1988) and Lucas (1991). Among them, one was proposed (earlier but then published) in Ormerod (2001), and it is based on interacting, heterogeneous agents that behave under uncertainties about the future but according to Keynesian principles. The model is quite general, and the rationale and good performance of the model are detailed in Chapter 9 of Ormerod (1998). Even though only partial solution of the model as “integral equations” has been provided in Ormerod (2001) or Ormerod (1998), neither the mathematical properties of the model nor how the individual agents’ growth rates that are tied by the model should behave has been analyzed. Further, the induced model for business cycles has not been analyzed mathematically. This makes understanding of the long term econometric behavior of the growth rates employed in the model and of the business cycles the induced model is able to capture, less transparent and somewhat difficult. To resolve these issues, we derive the explicit solutions to the model and the induced model, analyze the key properties of these solutions, and make connections between their mathematical features and econometric implications.

The rest of the article is organized as follows. In Section 2 we state the autoregressive model, and the explicit decomposition of the transition matrix (see (2.3)) involved in the model is provided in Section 3. The explicit solution of the model and its properties are given in Section 4, and solutions to the induced model for business cycles and their properties are explored in Section 5. A brief discussion in Section 6 ends the article.

2. THE VECTOR AUTOREGRESSIVE MODEL

For $i = 1, \dots, n$ with $n \in \mathbb{R}$ and $t \in \mathbb{Z}_+ = \{m \in \mathbb{Z} : m \geq 0\}$, let $x_i(t)$ be the growth rate of the output of the i ’th firm in period t and $y_i(t)$ the rate of change of the sentiment about the future of the i ’th firm formed in period t . Further, define

$$\mathcal{C}_n = \left\{ \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n : \min_{1 \leq i \leq n} w_i > 0, \sum_{i=1}^n w_i = 1 \right\}.$$

The overall rate of growth of the output is the weighted sum of the individual growth rates, defined as $\bar{x}(t) = \sum_{i=1}^n b_i x_i(t)$ for some $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{C}_n$, and the overall rate of growth of sentiment is the weighted sum of the individual $y_i(t)$, defined as $\bar{y}(t) = \sum_{i=1}^n a_i y_i(t)$ for some $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{C}_n$. Further, $\{x_i(t)\}_{i=1}^n$ and $\{y_i(t)\}_{i=1}^n$ are related by the model proposed in Ormerod (2001) as

$$(2.1) \quad \begin{cases} x_i(t+1) = (1 - \alpha) x_i(t) + \alpha [\bar{y}(t) + \varepsilon_i(t)] \\ y_i(t+1) = (1 - \beta) y_i(t) - \beta [\bar{x}(t) + \eta_i(t)] \end{cases}$$

for constants $\alpha, \beta \in \mathbb{R}$, $\varepsilon_i(t) \propto \mathbf{N}(\mu_i, \sigma_i^2)$ and $\eta_i(t) = \varepsilon_{n+i}(t) \propto \mathbf{N}(\mu_{n+i}, \sigma_{n+i}^2)$ for $\mu_i, \mu_{n+i} \in \mathbb{R}$ and σ_i and $\sigma_{n+i} > 0$, where $\xi \propto \mathbf{N}(\mu, \sigma^2)$ means that ξ is Normally distributed and has density

$$g_{\mu, \sigma^2}(x) = \left(\sqrt{2\pi\sigma} \right)^{-1} \exp \left[- (2\sigma^2)^{-1} (x - \mu)^2 \right]$$

for $x \in \mathbb{R}$. Here we assume that all random vectors are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, \mathcal{F} a sigma algebra on Ω , and \mathbb{P} the probability measure.

Let $\mathbf{x}_t = (x_1(t), \dots, x_n(t))$, $\mathbf{y}_t = (y_1(t), \dots, y_n(t))$, $\mathbf{z}_t = (\mathbf{x}_t, \mathbf{y}_t)^T$, $\boldsymbol{\varepsilon}_t = (\varepsilon_1(t), \dots, \varepsilon_n(t))$, $\boldsymbol{\eta}_t = (\eta_1(t), \dots, \eta_n(t))$, and $\boldsymbol{\gamma}_t = (\alpha\boldsymbol{\varepsilon}_t, -\beta\boldsymbol{\eta}_t)^T$, where the superscript T denote transpose of a matrix. Further, let $\mathcal{M}_{s \times s'}$ with $s, s' \in \mathbb{N}$ be the set of $s \times s'$ real matrices, which is denoted by \mathcal{M}_s when $s = s'$. Then model (2.1) can be rewritten as

$$(2.2) \quad \mathbf{z}_{t+1} = \mathbf{M}\mathbf{z}_t + \boldsymbol{\gamma}_t,$$

where the “transition matrix”

$$(2.3) \quad \mathbf{M} = \begin{pmatrix} (1-\alpha)\mathbf{I}_n & \alpha\mathbf{1}_n^T \mathbf{a} \\ -\beta\mathbf{1}_n^T \mathbf{b} & (1-\beta)\mathbf{I}_n \end{pmatrix} \in \mathcal{M}_{2n},$$

\mathbf{I}_s denotes the $s \times s$ identity matrix, and $\mathbf{1}_s$ is a row vector of s one's.

Model (2.2) bundles the agents' growth rates stored in the vector \mathbf{z}_t together, in which \mathbf{z}_t is sent into the very near future by the mapping induced by the matrix \mathbf{M} . Therefore, it put restrictions on how \mathbf{z}_t should behave jointly. However, no explicit solution in $\{\mathbf{z}_t\}_{\mathbb{Z}_+}$ has been available. This makes hard the direct simulation of $\{\mathbf{z}_t\}_{\mathbb{Z}_+}$ and difficult the understanding of the long term behavior of $\{\mathbf{z}_t\}_{\mathbb{Z}_+}$ both mathematically and econometrically.

3. DECOMPOSITION OF THE TRANSITION MATRIX

To probe into the long term behavior of $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ and how they are tied together in model (2.2), an efficient strategy is to decompose the $2n \times 2n$ matrix \mathbf{M} into products of simpler matrices. To this end, we need to understand the nontrivial invariant subspaces, if they exist, of the mapping induced by the $2n \times 2n$ matrix \mathbf{M} . To maintain good economic meaning of model (2.2), it is natural to assume

$$(3.1) \quad (\alpha, \beta) \notin \{(0, 0), (1, 1)\}.$$

The results to be presented in this section are independent of the distributional assumptions on $\boldsymbol{\gamma}_t$ for $t \in \mathbb{Z}_+$.

3.1. Jordan Canonical Form of \mathbf{M} . In this subsection, we provide the Jordan canonical form (see, e.g., [Jacobson \(1953\)](#) for a definition) of \mathbf{M} in the vector space \mathbb{R}^{2n} over \mathbb{R} . This will help convert the iterative identity, i.e., (2.2), for $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ into a direct, explicit representation in (4.1) without computing the averages \bar{x}_t or \bar{y}_t . For $\theta \in \mathbb{R}$ set $\mathbf{J}_1(\theta) = \theta$, and for a natural number $r \geq 2$ define the Jordan block

$$(3.2) \quad \mathbf{J}_r(\theta) = \begin{pmatrix} \theta & 1 & & & \\ & \theta & 1 & & \\ & & \ddots & \ddots & \\ & & & \theta & 1 \\ & & & & \theta \end{pmatrix} \in \mathcal{M}_r$$

whose diagonal entries are all θ , superdiagonal entries are all 1, and unmarked entries are identically zero. Let $f(\lambda) = |\lambda \mathbf{I} - \mathbf{M}|$ be the characteristic polynomial of \mathbf{M} , $\Delta = \alpha^2 + \beta^2 - 6\alpha\beta$, $d_1 = (3 - 2\sqrt{2})\beta$ and $d_2 = (3 + 2\sqrt{2})\beta$. The following theorem gives the roots of $f(\lambda)$ and conditions on if \mathbf{M} can be diagonalized.

Theorem 1. *The characteristic polynomial*

$$(3.3) \quad f(\lambda) = (\lambda - 1 + \beta)^{n-1} (\lambda - 1 + \alpha)^{n-1} g(\lambda)$$

with

$$(3.4) \quad g(\lambda) = (\lambda - 1)^2 + (\lambda - 1)(\alpha + \beta) + 2\alpha\beta.$$

So, $f(\lambda)$ always has real roots $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$. In addition, the following hold:

- (1) If $\min\{d_1, d_2\} < \alpha < \max\{d_1, d_2\}$, then $f(\lambda)$ has no other real roots and \mathbf{M} can not be diagonalized in the vector space \mathbb{R}^{2n} over \mathbb{R} .
- (2) If $\alpha \leq \min\{d_1, d_2\}$ or $\alpha \geq \max\{d_1, d_2\}$, then $f(\lambda)$ has two more real roots $\lambda_3 = 2^{-1}(2 - \alpha - \beta + \sqrt{\Delta})$ and $\lambda_4 = 2^{-1}(2 - \alpha - \beta - \sqrt{\Delta})$. If $\alpha < \min\{d_1, d_2\}$ or $\alpha > \max\{d_1, d_2\}$, the Jordan canonical form of \mathbf{M} is the diagonal matrix

$$(3.5) \quad \mathbf{J} = \mathbf{Q}^{-1}\mathbf{M}\mathbf{Q} = \text{diag}\{\lambda_1\mathbf{I}_{n-1}, \lambda_3\mathbf{I}_1, \lambda_2\mathbf{I}_{n-1}, \lambda_4\mathbf{I}_1\}$$

for some nonsingular matrix $\mathbf{Q} \in \mathcal{M}_{2n}$. However, if $\alpha = d_1$ or $\alpha = d_2$, \mathbf{M} can not be diagonalized in the vector space \mathbb{R}^{2n} over \mathbb{R} and the Jordan canonical form of \mathbf{M} is

$$(3.6) \quad \mathbf{J} = \mathbf{Q}^{-1}\mathbf{M}\mathbf{Q} = \text{diag}\{\lambda_1\mathbf{I}_{n-1}, \lambda_2\mathbf{I}_{n-1}, \mathbf{J}_2(\lambda_3)\}$$

for some nonsingular matrix $\mathbf{Q} \in \mathcal{M}_{2n}$.

Proof. For some $\varepsilon \geq 0$, let

$$(3.7) \quad \mathbf{M}_\varepsilon = \begin{pmatrix} (1 - \alpha - \varepsilon)\mathbf{I}_n & \alpha\mathbf{1}_n^T\mathbf{a} \\ -\beta\mathbf{1}_n^T\mathbf{b} & (1 - \beta)\mathbf{I}_n \end{pmatrix},$$

and

$$(3.8) \quad \mathbf{M}_{\lambda,\varepsilon} = \lambda\mathbf{I}_{2n} - \mathbf{M}_\varepsilon = \begin{pmatrix} (\lambda - 1 + \alpha + \varepsilon)\mathbf{I}_n & -\alpha\mathbf{1}_n^T\mathbf{a} \\ \beta\mathbf{1}_n^T\mathbf{b} & (\lambda - 1 + \beta)\mathbf{I}_n \end{pmatrix}.$$

Then $\mathbf{M}_0 = \mathbf{M}$, and it's obvious that $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$ are roots of $f(\lambda)$ since $\text{rank}(\mathbf{M}_{\lambda,0}) = n + 1 < 2n$ when $\lambda = \lambda_1$ or $\lambda = \lambda_2$. To find other roots of $f(\lambda)$, set

$$\mathbf{T}_\varepsilon = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \frac{-\beta}{\lambda - 1 + \alpha + \varepsilon}\mathbf{1}_n^T\mathbf{b} & \mathbf{I}_n \end{pmatrix} \in \mathcal{M}_{2n}$$

where $\varepsilon > 0$ is now assumed and $\mathbf{0}$ denotes a matrix with identical zero entries of compatible dimension. Then $|\mathbf{T}_\varepsilon| = 1$ and

$$\begin{aligned}
 \tilde{\mathbf{M}}_\varepsilon &= \mathbf{T}_\varepsilon \mathbf{M}_{\lambda, \varepsilon} \\
 &= \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \frac{-\beta}{\lambda-1+\alpha+\varepsilon} \mathbf{1}_n^T \mathbf{b} & \mathbf{I}_n \end{pmatrix} \begin{pmatrix} (\lambda-1+\alpha+\varepsilon) \mathbf{I} & -\alpha \mathbf{1}^T \mathbf{a} \\ \beta \mathbf{1}^T \mathbf{b} & (\lambda-1+\beta) \mathbf{I}_n \end{pmatrix} \\
 (3.9) \quad &= \begin{pmatrix} (\lambda-1+\alpha+\varepsilon) \mathbf{I}_n & -\alpha \mathbf{1}_n^T \mathbf{a} \\ \mathbf{0} & \frac{\alpha\beta}{\lambda-1+\alpha+\varepsilon} \mathbf{b} \mathbf{1}_n^T \mathbf{1}_n^T \mathbf{a} + (\lambda-1+\beta) \mathbf{I}_n \end{pmatrix}.
 \end{aligned}$$

Consequently, using the fact $\mathbf{b} \mathbf{1}_n^T = 1$ and Sylvester's determinant theorem, we obtain

$$\begin{aligned}
 f(\lambda) &= |\lambda \mathbf{I}_{2n} - \mathbf{M}| = \lim_{\varepsilon \rightarrow 0} |\mathbf{T}_\varepsilon^{-1}| |\tilde{\mathbf{M}}_\varepsilon| = \lim_{\varepsilon \rightarrow 0} |\tilde{\mathbf{M}}_\varepsilon| \\
 &= \lim_{\varepsilon \rightarrow 0} |(\lambda-1+\alpha+\varepsilon) \mathbf{I}_n| \left| \frac{\alpha\beta}{\lambda-1+\alpha+\varepsilon} \mathbf{1}_n^T \mathbf{a} + (\lambda-1+\beta) \mathbf{I}_n \right| \\
 &= \lim_{\varepsilon \rightarrow 0} (\lambda-1+\alpha+\varepsilon)^n (\lambda-1+\beta)^{n-1} \left[(\lambda-1+\beta) + \frac{\alpha\beta}{\lambda-1+\alpha+\varepsilon} \right]
 \end{aligned}$$

Thus, (3.3) and (3.4) hold, i.e.,

$$f(\lambda) = (\lambda-1+\beta)^{n-1} (\lambda-1+\alpha)^{n-1} g(\lambda).$$

This means that $f(\lambda)$ always has roots $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$.

Now we deal with the extra roots of $f(\lambda)$, for which the theory in Jacobson (1953) on Jordan canonical form will be applied. Without loss of generality (WLOG), assume for the rest of the proof that $d_1 = \min\{d_1, d_2\}$ and $d_2 = \max\{d_1, d_2\}$. It is easy to verify that the determinant of g in (3.4) is $\Delta = \alpha^2 + \beta^2 - 6\alpha\beta$ and that $\Delta = 0$ if and only if $\alpha = (3 - 2\sqrt{2})\beta$ or $\alpha = (3 + 2\sqrt{2})\beta$. By (3.3) and properties of quadratic functions, we see that f has two other real roots λ_3 and λ_4 when $\alpha \leq d_1$ or $\alpha \geq d_2$ but no more real roots when $d_1 < \alpha < d_2$. Since $f(\lambda)$ can not be written as $\prod_j (\lambda - \lambda_j)$ for reals λ_j when $d_1 < \alpha < d_2$, \mathbf{M} can not be diagonalized in the vector space \mathbb{R}^{2n} over \mathbb{R} .

Finally, we derive the Jordan blocks corresponding to each λ_i for $i = 1, \dots, 4$ when $\alpha \leq d_1$ or $\alpha \geq d_2$. In this case, $\alpha \neq \beta$, $f(\lambda)$ has the form $\prod_j (\lambda - \lambda_j)$ for real λ_j and \mathbf{M} can potentially be diagonalized. For $\lambda_1 = 1 - \alpha$, we have

$$(3.10) \quad \mathbf{M} - \lambda_1 \mathbf{I}_{2n} = \begin{pmatrix} \mathbf{0} & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & (\alpha - \beta) \mathbf{I}_n \end{pmatrix}.$$

So, when $\alpha \neq \beta$, we have $\text{rank}(\mathbf{M} - \lambda_1 \mathbf{I}_{2n}) = n + 1$ and

$$\rho_{\mathbf{M}}(\lambda_1) = 2n - \text{rank}(\mathbf{M} - \lambda_1 \mathbf{I}_{2n}) = n - 1,$$

where $\rho_{\mathbf{A}}(\lambda)$ denotes the dimension of the kernel space of $\mathbf{A} - \lambda \mathbf{I}_s$ for a square matrix $\mathbf{A} \in \mathcal{M}_s$ and $\lambda \in \mathbb{R}$ as a linear mapping $\mathbf{v} \mapsto (\mathbf{A} - \lambda \mathbf{I}_s) \mathbf{v}$ for a column vector $\mathbf{v} \in \mathbb{R}^s$.

For $\lambda_2 = 1 - \beta$, we have

$$(3.11) \quad \mathbf{M} - \lambda_2 \mathbf{I}_{2n} = \begin{pmatrix} (\beta - \alpha) \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & \mathbf{0} \end{pmatrix}.$$

So, when $\alpha \neq \beta$, $\text{rank}(\mathbf{M} - \lambda_2 \mathbf{I}_{2n}) = n + 1$ and

$$\rho_{\mathbf{M}}(\lambda_2) = 2n - \text{rank}(\mathbf{M} - \lambda_2 \mathbf{I}_{2n}) = n - 1.$$

For λ_3 and λ_4 when $\alpha < d_1$ or $\alpha > d_2$, we immediately see that the Jordan blocks corresponding to them are respectively $\mathbf{J}_1(\lambda_3) = \lambda_3$ and $\mathbf{J}_1(\lambda_4) = \lambda_4$ since each of λ_3 and λ_4 is a simple root of $f(\lambda)$. Thus, $\sum_{i=1}^4 \rho_{\mathbf{M}}(\lambda_i) = 2n$ and there is a nonsingular matrix \mathbf{Q} such that (3.5) holds.

However, for λ_3 and λ_4 when $\alpha = d_1$ or $\alpha = d_2$, $\lambda_3 = 1 - 2^{-1}(\alpha + \beta)$ becomes a double root and

$$\mathbf{M} - \lambda_3 \mathbf{I}_{2n} = \begin{pmatrix} 2^{-1}(\beta - \alpha) \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & -2^{-1}(\beta - \alpha) \mathbf{I}_n \end{pmatrix}.$$

In order to decide $\rho_{\mathbf{M}}(\lambda_3)$, the rank r_{λ_3} of $\mathbf{M} - \lambda_3 \mathbf{I}_{2n}$ needs to be obtained. From (3.9), we know that r_{λ_3} is that of

$$\mathbf{M}_{\lambda_3} = \begin{pmatrix} \frac{\alpha - \beta}{2} \mathbf{I}_n & -\alpha \mathbf{1}_n^T \mathbf{a} \\ \mathbf{0} & \frac{\beta - \alpha}{2} \mathbf{I}_n - \frac{2\alpha\beta}{\beta - \alpha} \mathbf{1}_n^T \mathbf{a} \end{pmatrix}.$$

Set $\mathbf{B} = \frac{\beta - \alpha}{2} \mathbf{I}_n - \frac{2\alpha\beta}{\beta - \alpha} \mathbf{1}_n^T \mathbf{a}$. Then $|\mathbf{B}| = 0$, i.e., $\text{rank}(\mathbf{B}) < n$ since $\mathbf{a} \mathbf{1}_n^T = 1$ and $\alpha = d_1$ or d_2 implies $\frac{\beta - \alpha}{2} = \frac{2\alpha\beta}{\beta - \alpha}$. So, it suffices to get the rank of \mathbf{B} to obtain r_{λ_3} . Let $\mathbf{a}_{(-1)}$ be the vector obtained by removing one entry from \mathbf{a} and $\mathbf{B}_{n-1} = \frac{\beta - \alpha}{2} \mathbf{I}_{n-1} - \frac{2\alpha\beta}{\beta - \alpha} \mathbf{1}_{n-1}^T \mathbf{a}_{(-1)}$. Then

$$\begin{aligned} |\mathbf{B}_{n-1}| &= \left| \frac{\beta - \alpha}{2} \left(\mathbf{I}_{n-1} - \frac{4\alpha\beta}{(\beta - \alpha)^2} \mathbf{1}_{n-1}^T \mathbf{a}_{(-1)} \right) \right| \\ &= \left(\frac{\beta - \alpha}{2} \right)^{n-1} (1 - \mathbf{a}_{(-1)} \mathbf{1}_{n-1}^T) \neq 0 \end{aligned}$$

by the definition of \mathbf{a} . Therefore, $\text{rank}(\mathbf{B}) = n - 1$ and $\rho_{\mathbf{M}}(\lambda_3) = 2n - \text{rank}(\mathbf{M} - \lambda_3 \mathbf{I}_{2n}) = 1$. This implies that the Jordan block corresponding to λ_3 is $\mathbf{J}_2(\lambda_3)$ and that $\sum_{i=1}^3 \rho_{\mathbf{M}}(\lambda_i) = 2n - 1$. Therefore, \mathbf{M} can not be diagonalized in the vector space \mathbb{R}^{2n} over \mathbb{R} . However, there exists a nonsingular $\mathbf{Q} \in \mathcal{M}_{2n}$ such that

$$\mathbf{J} = \mathbf{Q}^{-1} \mathbf{M} \mathbf{Q} = \text{diag} \{ \lambda_1 \mathbf{I}_{n-1}, \lambda_2 \mathbf{I}_{n-1}, \mathbf{J}_2(\lambda_3) \},$$

which justifies (3.6). This completes the proof. \square

3.2. Explicit Form of The Matrix of Basis. Theorem 1 provides an eigen decomposition of \mathbf{M} . However, it does not show what the matrix of basis \mathbf{Q} is. In what follows, we will only provide explicitly \mathbf{Q} for the second case in Theorem 1 for which \mathbf{M} can be diagonalized, since this case makes $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ the most amenable to an econometric analysis of its long term behavior. For an integer $s > 1$, let $\mathbf{e}_{i,s} \in \mathbb{R}^s$ be such that only the i th

entry of $\mathbf{e}_{i,s}$ is 1 but others are all zero, and for $\mathcal{B} \subseteq \mathbb{R}^s$ let $\text{span}(\mathcal{B})$ be the smallest linear space containing \mathcal{B} . We have:

Theorem 2. *Suppose $\alpha < d_1$ or $\alpha > d_2$ such that (3.5) holds, then the matrix \mathbf{Q} in (3.5) is given by*

$$(3.12) \quad \mathbf{Q} = \begin{pmatrix} \mathbf{W}_{\lambda_1}^T & \mathbf{W}_{\lambda_3}^T & \mathbf{W}_{\lambda_2}^T & \mathbf{W}_{\lambda_4}^T \end{pmatrix},$$

where, for $1 \leq i \leq n-1$,

$$(3.13) \quad \begin{cases} \mathbf{W}_{\lambda_1}^T = (\boldsymbol{\varepsilon}_1^T, \dots, \boldsymbol{\varepsilon}_{n-1}^T) \text{ with } \boldsymbol{\varepsilon}_i = (-b_1^{-1}b_{i+1}, \mathbf{e}_{i,n-1}, \mathbf{0}) \\ \mathbf{W}_{\lambda_2}^T = (\tilde{\boldsymbol{\varepsilon}}_1^T, \dots, \tilde{\boldsymbol{\varepsilon}}_{n-1}^T) \text{ with } \tilde{\boldsymbol{\varepsilon}}_i = (\mathbf{0}, -a_1^{-1}a_{i+1}, \mathbf{e}_{i,n-1}) \\ \mathbf{W}_{\lambda_3}^T = \left(\mathbf{1}_n, -2\alpha \left(\beta - \alpha - \sqrt{\Delta} \right)^{-1} \mathbf{1}_n \right)^T \\ \mathbf{W}_{\lambda_4}^T = \left(\mathbf{1}_n, -2\alpha \left(\beta - \alpha + \sqrt{\Delta} \right)^{-1} \mathbf{1}_n \right)^T. \end{cases}$$

Proof. Recall that $\mathbf{M}\mathbf{Q} = \mathbf{Q}\mathbf{J}$, where

$$\mathbf{J} = \text{diag} \{ \lambda_1, \dots, \lambda_1, \lambda_3, \lambda_2, \dots, \lambda_2, \lambda_4 \}$$

as in (3.5), and $\lambda_1 = 1 - \alpha$, $\lambda_2 = 1 - \beta$, $\lambda_{3,4} = 2^{-1} (2 - \alpha - \beta \pm \sqrt{\Delta})$. We will find \mathbf{Q} using the equations $\mathbf{M}\mathbf{Q} = \mathbf{Q}\mathbf{J}$ and (3.5) for each λ_i , $i = 1, \dots, 4$. To this end, let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{2n}$ with $\mathbf{x}_1 = (x_1, \dots, x_n)$ and $\mathbf{x}_2 = (x_{n+1}, \dots, x_{2n})$.

For $\lambda_1 = 1 - \alpha$, we have

$$\mathbf{M} - \lambda_1 \mathbf{I}_{2n} = \begin{pmatrix} \mathbf{0} & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & (\alpha - \beta) \mathbf{I}_n \end{pmatrix}$$

and that $(\mathbf{M} - \lambda_1 \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$ if and only if

$$(3.14) \quad \mathbf{a} \mathbf{x}_2^T = 0 \text{ and } \mathbf{x}_2^T = \frac{\beta \mathbf{b} \mathbf{x}_1^T}{\alpha - \beta} \mathbf{1}_n^T.$$

Since $\mathbf{a} \in \mathcal{C}_n$ and $\mathbf{a} \mathbf{x}_2^T = 0$, it is clear that when $\mathbf{x}_2^T \neq \mathbf{0}$ some component of \mathbf{x}_2 have to be positive and some negative. However, since the sign of $\frac{\beta \mathbf{b} \mathbf{x}_1^T}{\alpha - \beta}$ is fixed, the second identity in (3.14) thus forces the components of \mathbf{x}_2 to have the same sign. Therefore, (3.14) holds if and only if $\mathbf{x}_2^T = \mathbf{0}$, and this gives $\mathbf{b} \mathbf{x}_1^T = 0$. In other words,

$$(3.15) \quad \tilde{\mathbf{W}}_{\lambda_1} = \ker(\mathbf{M} - \lambda_1 \mathbf{I}_{2n}) = \{ \mathbf{x} \in \mathbb{R}^{2n} : \mathbf{b} \mathbf{x}_1^T = 0, \mathbf{x}_2^T = \mathbf{0} \},$$

where $\ker(\mathbf{A})$ denotes the kernel space of a square matrix $\mathbf{A} \in \mathcal{M}_s$ as a linear mapping $\mathbf{v} \mapsto \mathbf{A}\mathbf{v}$ for a column vector $\mathbf{v} \in \mathbb{R}^s$. Since $\dim(\tilde{\mathbf{W}}_{\lambda_1}) = n-1$, we see that the eigenspace corresponding to λ_1 is $\tilde{\mathbf{W}}_{\lambda_1}$. Further, it is easy to verify that $\boldsymbol{\varepsilon}_i = (-b_1^{-1}b_{i+1}, \mathbf{e}_{i,n-1}, \mathbf{0}) \in \mathbb{R}^{2n}$ for $1 \leq i \leq n-1$ is a basis for $\tilde{\mathbf{W}}_{\lambda_1}$.

For $\lambda_2 = 1 - \beta$, we have

$$\mathbf{M} - \lambda_2 \mathbf{I}_{2n} = \begin{pmatrix} (\beta - \alpha) \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & \mathbf{0} \end{pmatrix}.$$

So, $(\mathbf{M} - \lambda_2 \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$ if and only if

$$(3.16) \quad \mathbf{b} \mathbf{x}_1^T = 0 \text{ and } \mathbf{x}_1^T = \frac{\alpha \mathbf{a} \mathbf{x}_2^T}{\alpha - \beta} \mathbf{1}_n^T.$$

By the same reasoning for the case of λ_1 , we see that

$$(3.17) \quad \tilde{\mathbf{W}}_{\lambda_2} = \ker(\mathbf{M} - \lambda_2 \mathbf{I}_{2n}) = \{ \mathbf{x} \in \mathbb{R}^{2n} : \mathbf{a} \mathbf{x}_2^T = 0, \mathbf{x}_1^T = \mathbf{0} \}$$

with $\dim(\tilde{\mathbf{W}}_{\lambda_1}) = n - 1$ is the eigenspace corresponding to λ_3 . Further, it is easy to verify that $\tilde{\mathbf{e}}_i = (\mathbf{0}, -a_1^{-1} a_{i+1}, \mathbf{e}_{i,n-1}) \in \mathbb{R}^{2n}$ for $1 \leq i \leq n - 1$ is a basis for $\tilde{\mathbf{W}}_{\lambda_2}$.

For $\lambda_{3,4} = 1 - 2^{-1}(\alpha + \beta) \pm 2^{-1}\sqrt{\Delta}$, we have

$$\mathbf{M} - \lambda_{3,4} \mathbf{I}_{2n} = \begin{pmatrix} \tau \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & \delta \mathbf{I}_n \end{pmatrix},$$

where $\tau = \frac{\beta - \alpha}{2} \mp \frac{\sqrt{\Delta}}{2}$ and $\delta = \frac{\alpha - \beta}{2} \mp \frac{\sqrt{\Delta}}{2}$. Let

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \tau^{-1} \beta \mathbf{1}_n^T \mathbf{b} & \mathbf{I}_n \end{pmatrix}$$

and $\tilde{\mathbf{M}}_2 = \mathbf{T}_2 (\mathbf{M} - \lambda_{3,4} \mathbf{I}_{2n})$. Then

$$\tilde{\mathbf{M}}_2 = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \tau^{-1} \beta \mathbf{1}_n^T \mathbf{b} & \mathbf{I}_n \end{pmatrix} \begin{pmatrix} \tau \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ -\beta \mathbf{1}_n^T \mathbf{b} & \delta \mathbf{I}_n \end{pmatrix} = \begin{pmatrix} \tau \mathbf{I}_n & \alpha \mathbf{1}_n^T \mathbf{a} \\ \mathbf{0} & \tau^{-1} \beta \alpha \mathbf{1}_n^T \mathbf{a} + \delta \mathbf{I}_n \end{pmatrix},$$

and $\ker(\mathbf{M} - \lambda_{3,4} \mathbf{I}_{2n}) = \ker(\tilde{\mathbf{M}}_2)$ since $|\mathbf{T}_2| = 1$. Obviously, $\tilde{\mathbf{M}}_2 \mathbf{x}^T = \mathbf{0}$ if and only if

$$(3.18) \quad \tau \mathbf{x}_1^T + \alpha \mathbf{1}_n^T \mathbf{a} \mathbf{x}_2^T = \mathbf{0} \quad \text{and} \quad (\tau^{-1} \delta^{-1} \beta \alpha \mathbf{1}_n^T \mathbf{a} + \mathbf{I}_n) \mathbf{x}_2^T = \mathbf{0}.$$

Since

$$\tau \delta = -4^{-1} (\beta - \alpha \mp \sqrt{\Delta}) (\beta - \alpha \pm \sqrt{\Delta}) = -4^{-1} [(\beta - \alpha)^2 - \Delta] = -\alpha \beta$$

and $\tau^{-1} \delta^{-1} \beta \alpha = -1$, the second identity in (3.18) becomes

$$(3.19) \quad (\mathbf{1}_n^T \mathbf{a} - \mathbf{I}_n) \mathbf{x}_2^T = \mathbf{0}$$

Since the matrix $\mathbf{a} \mathbf{1}_n^T = \mathbf{1}$ has the only eigenvalue 1 whose corresponding eigenvector is $\mathbf{1}$, the general solution to (3.19) is $\mathbf{x}_2^T = c \mathbf{1}_n^T$ for some $c \in \mathbb{R}$. Let $\mathbf{R} = \mathbf{I}_{n-1} - \mathbf{1}_{n-1}^T \mathbf{a}_{(-n)}$, where $\mathbf{a}_{(-n)} = (a_1, \dots, a_{n-1})$. Then $|\mathbf{R}| = 1 - \sum_{i=1}^{n-1} a_i > 0$ since $\mathbf{a} \in \mathcal{C}_n$ and $\text{rank}(\mathbf{R}) = n - 1$. So, $\ker(\mathbf{1}_n^T \mathbf{a} - \mathbf{I}_n) = \text{span}(\{\mathbf{1}_n\})$, and the general solution to $(\mathbf{M} - \lambda_{3,4} \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$ is

$$(3.20) \quad \mathbf{x}_1 = -\alpha \tau^{-1} \mathbf{x}_2 \text{ and } \mathbf{x}_2 = c \mathbf{1}_n.$$

Therefore, the solution space to $(\mathbf{M} - \lambda_3 \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$, i.e., that when $\tau = \frac{\beta - \alpha - \sqrt{\Delta}}{2}$ and $\delta = \frac{\alpha - \beta - \sqrt{\Delta}}{2}$, is

$$(3.21) \quad \tilde{\mathbf{W}}_{\lambda_3} = \text{span} \left(\left\{ \mathbf{x} \in \mathbb{R}^{2n} : \mathbf{x} = \left(\mathbf{1}_n, -2\alpha (\beta - \alpha - \sqrt{\Delta})^{-1} \mathbf{1}_n \right) \right\} \right).$$

Further, the solution space to $(\mathbf{M} - \lambda_4 \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$, i.e., that when $\tau = \frac{\beta - \alpha + \sqrt{\Delta}}{2}$ and $\delta = \frac{\alpha - \beta + \sqrt{\Delta}}{2}$, is

$$(3.22) \quad \tilde{\mathbf{W}}_{\lambda_4} = \text{span} \left(\left\{ \mathbf{x} \in \mathbb{R}^{2n} : \mathbf{x} = \begin{pmatrix} \mathbf{1}_n, -2\alpha (\beta - \alpha + \sqrt{\Delta})^{-1} \mathbf{1}_n \end{pmatrix} \right\} \right).$$

Combining the solutions to equations $(\mathbf{M} - \lambda_i \mathbf{I}_{2n}) \mathbf{x}^T = \mathbf{0}$ for $1 \leq i \leq 4$, we see that (3.12) holds with (3.13). This completes the proof. \square

3.3. Inverse of The Matrix for Basis. Next we derive the inverse \mathbf{Q}^{-1} of \mathbf{Q} so that the full, explicit decomposition of \mathbf{M} will be available. Even though it is difficult in general to find explicitly the inverse of a large-dimensional matrix, i.e., when n is large or equivalently there are many growth rates involved in model (2.2), the inverse \mathbf{Q}^{-1} turns out to be very simple (see Theorem 3) due to the fact that the weights \mathbf{a} and \mathbf{b} both represent convex combinations and lie in the simplex \mathcal{C}_n .

In order to state the result, we introduce some notations. Let $\tau_- = 2 (\beta - \alpha - \sqrt{\Delta})^{-1}$, $\tau_+ = 2 (\beta - \alpha + \sqrt{\Delta})^{-1}$, $\mathbf{a}_{(-1)} = (a_2, \dots, a_n)$ and $\mathbf{b}_{(-1)} = (b_2, \dots, b_n)$. Recall (3.12) and (3.13). Then \mathbf{Q} can be written into a 4×4 block matrix as

$$(3.23) \quad \mathbf{Q} = \begin{pmatrix} -b_1^{-1} \mathbf{b}_{(-1)} & 1 & \mathbf{0}_{n \times (n-1)} & \mathbf{1}_n^T \\ \mathbf{I}_{n-1} & \mathbf{1}_{n-1}^T & & \\ \mathbf{0}_{n \times (n-1)} & -\alpha \tau_- \mathbf{1}_n^T & -a_1^{-1} \mathbf{a}_{(-1)} & -\alpha \tau_+ \\ & & \mathbf{I}_{n-1} & -\alpha \tau_+ \mathbf{1}_{n-1}^T \end{pmatrix},$$

where $\mathbf{0}_{s \times s'} \in \mathcal{M}_{s \times s'}$ has all entries as zero. Further, for integers i and j such that $1 \leq i \leq j \leq 2n$, let $\mathbf{E}_{ij} \in \mathcal{M}_{2n}$ be such that its ij th entry is 1 and other entries are identically zero, and for $\hat{c} \in \mathbb{R}$ let $\mathbf{P}_{i,j}(\hat{c}) = \mathbf{I}_{2n} + \hat{c} \mathbf{E}_{i,j}$. Note that for any matrix $\mathbf{A} = (\tilde{a}_{ij}) \in \mathcal{M}_{2n}$ the j th column of $\mathbf{A} \mathbf{E}_{ij}$ is the i th column of \mathbf{A} and all other entries of $\mathbf{A} \mathbf{E}_{ij}$ are zero.

Theorem 3. Under the conditions of Theorem 2, the inverse \mathbf{Q}^{-1} of \mathbf{Q} is

$$(3.24) \quad \mathbf{Q}^{-1} = \mathbf{P}_{n,2n}(-1) \mathbf{P}_{2n,n} \mathbf{P}_{2n,n}(\tilde{\tau}^{-1} \alpha \tau_-) \text{diag} \{ \mathbf{Q}_{21}^{-1}, \mathbf{Q}_{22}^{-1} \},$$

where $\tilde{\tau} = \alpha(\tau_- - \tau_+)$,

$$(3.25) \quad \mathbf{Q}_{21}^{-1} = \begin{pmatrix} -b_1 \mathbf{1}_{n-1}^T & \mathbf{I}_{n-1} - \mathbf{1}_{n-1}^T \mathbf{b}_{(-1)} \\ b_1 & \mathbf{b}_{(-1)} \end{pmatrix}$$

and

$$(3.26) \quad \mathbf{Q}_{22}^{-1} = \begin{pmatrix} -a_1 \mathbf{1}_{n-1}^T & \mathbf{I}_{n-1} - \mathbf{1}_{n-1}^T \mathbf{a}_{(-1)} \\ \tilde{\tau}^{-1} a_1 & \tilde{\tau}^{-1} \mathbf{a}_{(-1)} \end{pmatrix}.$$

Proof. Multiplying the n -th column of \mathbf{Q} by -1 and adding the resultant column to the $2n$ -th column of \mathbf{Q} gives

$$\mathbf{Q}_1 = \mathbf{Q} \mathbf{P}_{n,2n}(-1) = \begin{pmatrix} -b_1^{-1} \mathbf{b}_{(-1)} & 1 & \mathbf{0}_{n \times n} \\ \mathbf{I}_{n-1} & \mathbf{1}_{n-1}^T & \\ \mathbf{0}_{n \times (n-1)} & -\alpha \tau_- \mathbf{1}_n^T & -a_1^{-1} \mathbf{a}_{(-1)} & \tilde{\tau} \\ & & \mathbf{I}_{n-1} & \tilde{\tau} \mathbf{1}_{n-1}^T \end{pmatrix}$$

with $\tilde{\tau} = \alpha(\tau_- - \tau_+)$. Multiplying the $2n$ -th column of \mathbf{Q}_1 by $\tilde{\tau}^{-1} \alpha \tau_-$ and adding the resultant column to the n -th column of \mathbf{Q}_1 gives

$$\mathbf{Q}_2 = \mathbf{Q}_1 \mathbf{P}_{2n,n}(-\Delta^{-1/2} \alpha \tau_-) = \begin{pmatrix} -b_1^{-1} \mathbf{b}_{(-1)} & 1 & \mathbf{0}_{n \times n} \\ \mathbf{I}_{n-1} & \mathbf{1}_{n-1}^T & \\ \mathbf{0}_{n \times n} & & -a_1^{-1} \mathbf{a}_{(-1)} & \tilde{\tau} \\ & & \mathbf{I}_{n-1} & \tilde{\tau} \mathbf{1}_{n-1}^T \end{pmatrix}.$$

Let

$$(3.27) \quad \mathbf{Q}_{21} = \begin{pmatrix} -b_1^{-1} \mathbf{b}_{(-1)} & 1 \\ \mathbf{I}_{n-1} & \mathbf{1}_{n-1}^T \end{pmatrix} \text{ and } \mathbf{Q}_{22} = \begin{pmatrix} -a_1^{-1} \mathbf{a}_{(-1)} & \tilde{\tau} \\ \mathbf{I}_{n-1} & \tilde{\tau} \mathbf{1}_{n-1}^T \end{pmatrix}.$$

Then $\mathbf{Q}_2 = \text{diag}\{\mathbf{Q}_{21}, \mathbf{Q}_{22}\}$ and

$$(3.28) \quad \mathbf{Q}^{-1} = \mathbf{P}_{n,2n}(-1) \mathbf{P}_{2n,n} \mathbf{P}_{2n,n}(\tilde{\tau}^{-1} \alpha \tau_-) \mathbf{Q}_2^{-1}.$$

Therefore, it suffices to find $\mathbf{Q}_2^{-1} = \text{diag}\{\mathbf{Q}_{21}^{-1}, \mathbf{Q}_{22}^{-1}\}$ or equivalently to find \mathbf{Q}_{21}^{-1} and \mathbf{Q}_{22}^{-1} .

Let

$$\mathbf{R}_1 = \begin{pmatrix} \mathbf{0}_{(n-1) \times 1} & \mathbf{I}_{n-1} \\ 1 & \mathbf{0}_{1 \times (n-1)} \end{pmatrix}.$$

Then $\mathbf{R}_1^{-1} = \mathbf{R}_1^T$, left multiplication by \mathbf{R}_1 permutes the rows, and right multiplication by \mathbf{R}_1 permutes the columns. Further,

$$\tilde{\mathbf{Q}}_{21} = \mathbf{R}_1 \mathbf{Q}_{21} = \begin{pmatrix} \mathbf{I}_{n-1} & \mathbf{1}_{n-1}^T \\ -b_1^{-1} \mathbf{b}_{(-1)} & 1 \end{pmatrix}.$$

Since $1 + b_1^{-1} \mathbf{b}_{(-1)} \mathbf{1}_{n-1}^T = b_1^{-1} \neq 0$, we have

$$(3.29) \quad \tilde{\mathbf{Q}}_{21}^{-1} = \begin{pmatrix} \mathbf{I}_{n-1} - \mathbf{1}_{n-1}^T \mathbf{b}_{(-1)} & -b_1 \mathbf{1}_{n-1}^T \\ \mathbf{b}_{(-1)} & b_1 \end{pmatrix}$$

and $\mathbf{Q}_{21}^{-1} = \tilde{\mathbf{Q}}_{21}^{-1} \mathbf{R}_1$, which implies (3.25). To get \mathbf{Q}_{22}^{-1} , we start from

$$\tilde{\mathbf{Q}}_{22} = \mathbf{R}_1 \mathbf{Q}_{22} = \begin{pmatrix} \mathbf{I}_{n-1} & \tilde{\tau} \mathbf{1}_{n-1}^T \\ -a_1^{-1} \mathbf{a}_{(-1)} & \tilde{\tau} \end{pmatrix}.$$

Since $\tilde{\tau} + a_1^{-1} \mathbf{a}_{(-1)} \tilde{\tau} \mathbf{1}_{n-1}^T = \tilde{\tau} a_1^{-1} \neq 0$, we see

$$(3.30) \quad \tilde{\mathbf{Q}}_{22}^{-1} = \begin{pmatrix} \mathbf{I}_{n-1} - \mathbf{1}_{n-1}^T \mathbf{a}_{(-1)} & -a_1 \mathbf{1}_{n-1}^T \\ \tilde{\tau}^{-1} \mathbf{a}_{(-1)} & \tilde{\tau}^{-1} a_1 \end{pmatrix}$$

and $\mathbf{Q}_{22}^{-1} = \tilde{\mathbf{Q}}_{22}^{-1} \mathbf{R}_1$, which implies (3.26).

Combining (3.25), (3.26) and (3.28), we get (3.24), which completes the proof. \square

Theorem 3 shows that \mathbf{Q}^{-1} has an easy and explicit form that allows quick computation even when n is large, since \mathbf{Q}_{21}^{-1} and \mathbf{Q}_{22}^{-1} are very simple and $\mathbf{P}_{n,2n}(-1)$ and $\mathbf{P}_{2n,n}(\tilde{\tau}^{-1}\alpha\tau_-)$ are only two linear operations on two columns of $\text{diag}\{\mathbf{Q}_{21}^{-1}, \mathbf{Q}_{22}^{-1}\}$. The inverse \mathbf{Q}^{-1} helps give the explicit decomposition of \mathbf{M} and the explicit solution $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ in **Corollary 1** that reveals its long term behavior.

4. THE EXPLICIT SOLUTION AND ITS PROPERTIES

We are ready to provide the explicit solution $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ to model (2.2) using the explicit decomposition of \mathbf{M} in terms of \mathbf{J} , \mathbf{Q} and \mathbf{Q}^{-1} given in **Theorem 1**, **Theorem 2**, and **Theorem 3**.

Corollary 1. *Under the conditions of **Theorem 2**, model (2.2) has the explicit solution*

$$(4.1) \quad \mathbf{z}_{t+1} = \mathbf{Q}\mathbf{J}^{t+1}\mathbf{Q}^{-1}\mathbf{z}_0 + \sum_{i=0}^t \mathbf{Q}\mathbf{J}^i\mathbf{Q}^{-1}\gamma_{t-i}$$

and explicit, equivalent solution

$$(4.2) \quad \tilde{\mathbf{z}}_{t+1} = \mathbf{J}^{t+1}\tilde{\mathbf{z}}_0 + \sum_{i=0}^t \mathbf{J}^i\tilde{\gamma}_{t-i},$$

where $\tilde{\mathbf{z}}_t = \mathbf{Q}^{-1}\mathbf{z}_t$ and $\tilde{\gamma}_t = \mathbf{Q}^{-1}\gamma_t$ for $t \in \mathbb{Z}_+$.

Proof. By results in **Section 3**, model (2.2) is just

$$\mathbf{z}_{t+1} = \mathbf{M}^{t+1}\mathbf{z}_0 + \sum_{i=0}^t \mathbf{M}^i\gamma_{t-i}$$

with the initial value \mathbf{z}_0 , where \mathbf{Q} is given in (3.12), \mathbf{J} in (3.5), and \mathbf{Q}^{-1} in (3.24). This implies (4.1) and (4.2), and completes the proof. \square

In other words, $\{\mathbf{z}_t\}_{t \in \mathbb{Z}}$ can be represented almost as a vector moving average model of order $t - 1$ with independent and identically distributed (i.i.d.) errors $\{\gamma_t\}_{t \in \mathbb{Z}}$.

4.1. Nonstationarity. Recall that a stochastic process is second-order stationary if its covariance function of a fixed lag depends only on the lag but not on the time index. In order to study the behavior of $\{\mathbf{z}_t\}_{t \in \mathbb{Z}}$, we need the following lemma on equivalence of second-order stationarity.

Lemma 1. *Under the conditions of **Theorem 2**, both or neither of the sequence $\{\tilde{\mathbf{z}}_t\}_{t \in \mathbb{Z}_+}$ defined in **Corollary 1** and $\{\mathbf{z}_t\}_{t \in \mathbb{Z}_+}$ are second-order stationary.*

Proof. Let

$$(4.3) \quad \mathbf{\Gamma}_{t+\tau',t} = \mathbb{E} \left[(\mathbf{z}_{t+\tau'} - \mathbb{E}(\mathbf{z}_{t+\tau'})) (\mathbf{z}_t - \mathbb{E}(\mathbf{z}_t))^T \right]$$

for $t, \tau' \in \mathbb{Z}_+$, where \mathbb{E} the expectation with respect to \mathbb{P} . Then $\mathbb{E}(\tilde{z}_t) = \mathbf{Q}^{-1}\mathbb{E}(z_t)$ and

$$(4.4) \quad \tilde{\mathbf{\Gamma}}_{t+\tau',t} = \mathbb{E} \left[(\tilde{z}_{t+\tau'} - \mathbb{E}(\tilde{z}_t)) (\tilde{z}_{t+\tau'} - \mathbb{E}(\tilde{z}_t))^T \right] = \mathbf{Q}^{-1} \mathbf{\Gamma}_{t+\tau',t} (\mathbf{Q}^{-1})^T.$$

This, together with the nonsingularity of \mathbf{Q} , implies that either both or neither $\{\tilde{z}_t\}_{t \in \mathbb{Z}_+}$ and $\{z_t\}_{t \in \mathbb{Z}_+}$ are second-order stationary. This completes the proof. \square

By Lemma 1, it suffices to study the second-order stationarity of $\{\tilde{z}_t\}_{t \in \mathbb{Z}}$. By the assumptions on $\{\varepsilon_i(t)\}_{i=1}^{2n}$ given in Section 1, we have the mean vector of γ_t as

$$\mu_t = (\alpha\mu_1, \dots, \alpha\mu_n, -\beta\mu_{n+1}, \dots, -\beta\mu_{2n})^T$$

and the covariance matrix Σ_t of γ_t as

$$\Sigma_t = \text{diag} \{ \alpha^2 \sigma_1^2, \dots, \alpha^2 \sigma_n^2, \beta^2 \sigma_{n+1}^2, \dots, \beta^2 \sigma_{2n}^2 \}.$$

The following result shows that $\{z_t\}_{t \in \mathbb{Z}_+}$ is not second-order stationary.

Proposition 1. *Suppose z_0 is independent of the sequence $\{\gamma_t\}_{t \in \mathbb{Z}_+}$ and has covariance matrix \mathbf{G} . Then $\tilde{\mathbf{\Gamma}}_{t+\tau',t}$ in (4.4) for $t \geq 2$ satisfies*

$$(4.5) \quad \tilde{\mathbf{\Gamma}}_{t+\tau',t} = \mathbf{J}^{t+\tau'} \mathbf{G} \mathbf{J}^t + \sum_{i=1}^{t-1} \mathbf{J}^{t'+i} \Sigma_0 \mathbf{J}^i.$$

Therefore, neither $\{z_t\}_{t \in \mathbb{Z}}$ nor $\{\tilde{z}_t\}_{t \in \mathbb{Z}_+}$ is second-order stationary.

Proof. To compute $\tilde{\mathbf{\Gamma}}_{t+\tau',t}$, it suffices to assume that z_0 and each $\gamma_t, t \in \mathbb{Z}_+$ has mean zero but with their corresponding covariances. Namely, it suffices to assume

$$\tilde{z}_{t+1} = \mathbf{J}^{t+1} \hat{z}_0 + \sum_{i=0}^t \mathbf{J}^i \hat{\gamma}_{t-i},$$

where \hat{z}_0 is the mean centered z_0 and each $\hat{\gamma}_t$ is the mean centered γ_t . This implies that

$$\tilde{\mathbf{\Gamma}}_{t+\tau',t} = \mathbb{E} [\tilde{z}_{t+\tau'} \tilde{z}_t^T] = \mathbb{E} \left[\left(\mathbf{J}^{t+\tau'} \hat{z}_0 + \sum_{i=0}^{t+\tau'-1} \mathbf{J}^i \hat{\gamma}_{t-i} \right) \left(\mathbf{J}^t \hat{z}_0 + \sum_{i=0}^{t-1} \mathbf{J}^i \hat{\gamma}_{t-i} \right)^T \right],$$

which simplifies into (4.5). Since $\tilde{\mathbf{\Gamma}}_{t+\tau',t}$ depends on t , $\{\tilde{z}_t\}_{t \in \mathbb{Z}_+}$ is not second-order stationary, and by Lemma 1 nor is $\{z_t\}_{t \in \mathbb{Z}_+}$. This completes the proof. \square

4.2. Limiting Behavior. For the representations given in (4.1) and (4.2), it is easy to explore the long term behavior of $\{\tilde{z}_t\}_{t \in \mathbb{Z}_+}$ than that of $\{z_t\}_{t \in \mathbb{Z}}$. Recall

$$\mathbf{J} = \text{diag} \{ \lambda_1 \mathbf{I}_{n-1}, \lambda_3 \mathbf{I}_1, \lambda_2 \mathbf{I}_{n-1}, \lambda_4 \mathbf{I}_1 \}$$

for which $\lambda_1 = 1 - \alpha$, $\lambda_2 = 1 - \beta$, $\lambda_{3,4} = 2^{-1} (2 - \alpha - \beta \pm \sqrt{\Delta})$.

Corollary 2. *Under the conditions of Theorem 2, if moreover*

$$(4.6) \quad 0 < \max_{1 \leq i \leq 4} |\lambda_i| < 1,$$

then, as $t \rightarrow \infty$,

$$(4.7) \quad \tilde{\mathbf{z}}_{t+1} \xrightarrow{\mathcal{D}} \text{diag} \left\{ \tilde{\lambda}_1 \mathbf{I}_{n-1}, \tilde{\lambda}_3 \mathbf{I}_1, \tilde{\lambda}_2 \mathbf{I}_{n-1}, \tilde{\lambda}_4 \mathbf{I}_1 \right\} \tilde{\gamma}_0$$

and

$$\mathbf{z}_{t+1} \xrightarrow{\mathcal{D}} \mathbf{Q} \text{diag} \left\{ \tilde{\lambda}_1 \mathbf{I}_{n-1}, \tilde{\lambda}_3 \mathbf{I}_1, \tilde{\lambda}_2 \mathbf{I}_{n-1}, \tilde{\lambda}_4 \mathbf{I}_1 \right\} \mathbf{Q}^{-1} \gamma_0,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $\tilde{\lambda}_i = (1 - \lambda_i)^{-1}$ for $1 \leq i \leq 4$.

The proof of [Corollary 2](#) is omitted as it follows immediately from the convergence of $\sum_{i=0}^{\infty} \lambda_j^i = (1 - \lambda_j)^{-1} \neq 0$ for $1 \leq j \leq 4$ when (4.6) holds, the fact that $\tilde{\mathbf{z}}_t = \mathbf{Q}^{-1} \mathbf{z}_t$ and $\tilde{\gamma}_t = \mathbf{Q}^{-1} \gamma_t$ for $t \in \mathbb{Z}_+$, and the i.i.d. property of $\{\tilde{\gamma}_t\}_{t \in \mathbb{Z}_+}$. It is clear that any $\lambda_i > 1$ for some $1 \leq i \leq 4$ lead to the explosion in the variance of the corresponding subvector of $\tilde{\mathbf{z}}_t$ and that of some subvector of \mathbf{z}_t as $t \rightarrow \infty$. Further, when $\lambda_i, 1 \leq i \leq 4$ have different signs, oscillations in components of $\tilde{\mathbf{z}}_t$ and those of \mathbf{z}_t will be induced. However, a complete analysis of such oscillations seems to be difficult to perform for the original series $\{\mathbf{z}_t\}_{t \in \mathbb{Z}}$ due to the term $\mathbf{J}^i \mathbf{Q}^{-1} \gamma_{t-i}$ in (4.1).

5. THE INDUCED MODEL FOR BUSINESS CYCLES

Recall that the aim of proposing model (2.2) in [Ormerod \(2001\)](#) is to induce a model for business cycles in the economy created by the growth rates $x_i(t)$ under the influence of the sentiments $y_i(t)$. Such an induced model has been given in [Ormerod \(2001\)](#) but with a miscalculated forcing term (see identity (10.9A) therein) and is called a “damped pendulum”. Despite its being called so, the induced model for business cycles is similar to an autoregressive (AR) model of order 2 in \bar{x}_t for $t \in \mathbb{Z}_+$, which we now provide and analyze.

From model (2.2), we have, by weighting corresponding with \mathbf{a} and \mathbf{b} ,

$$(5.1) \quad \begin{cases} \bar{x}(t+1) = (1 - \alpha) \bar{x}(t) + \alpha \bar{y}(t) + \alpha \bar{\varepsilon}(t), \\ \bar{y}(t+1) = (1 - \beta) \bar{y}(t) - \beta \bar{x}(t) - \beta \bar{\eta}(t), \end{cases}$$

where $\bar{\varepsilon}(t) = \sum_{i=1}^n b_i \varepsilon_i(t)$ and $\bar{\eta}(t) = \sum_{i=1}^n a_i \eta_i(t)$. From the first identity in (5.1), we obtain

$$(5.2) \quad \bar{y}(t) = \alpha^{-1} [\bar{x}(t+1) - (1 - \alpha) \bar{x}(t) - \alpha \bar{\varepsilon}(t)] = \alpha^{-1} \Delta \bar{x}(t) + \bar{x}(t) - \bar{\varepsilon}(t),$$

where $\Delta \bar{x}(t) = \bar{x}(t+1) - \bar{x}(t)$. Plugging (5.2) back into the second identity of (5.1), we get

$$(5.3) \quad \begin{aligned} & \alpha^{-1} \Delta \bar{x}(t+1) - \alpha^{-1} \Delta \bar{x}(t) + \bar{x}(t+1) - \bar{x}(t) + \beta \alpha^{-1} \Delta \bar{x}(t) + 2\beta \bar{x}(t) \\ &= \bar{\varepsilon}(t+1) - (1 - \beta) \bar{\varepsilon}(t) - \beta \bar{\eta}(t). \end{aligned}$$

After simplification, (5.3) becomes

$$(5.4) \quad \Delta^2 \bar{x}(t) + (\alpha + \beta) \Delta \bar{x}(t) + 2\alpha\beta \bar{x}(t) = h(t),$$

where $\Delta^2 \bar{x}(t) = \Delta \bar{x}(t+1) - \Delta \bar{x}(t)$, $\Delta \bar{\varepsilon}(t) = \bar{\varepsilon}(t+1) - \bar{\varepsilon}(t)$ and

$$(5.5) \quad h(t) = \alpha \Delta \bar{\varepsilon}(t) + \alpha \beta [\bar{\varepsilon}(t) - \bar{\eta}(t)].$$

Equations (5.4) and (5.5) together describe what is called in Ormerod (2001) a “damped pendulum”, for which $h(t)$ is the forcing term. Note however that $h(t)$ in (5.5), the correct one, is different than the mistaken one in identity (10.9A) therein.

5.1. The Periodic Solution. On the other hand, (5.4) and (5.5) almost form a second order difference equation in \bar{x}_t for $t \in \mathbb{Z}_+$ except that the random error $h(t)$ involves a term at time $t+1$. Specifically,

$$(5.6) \quad \bar{x}(t+2) + (\alpha + \beta - 2) \bar{x}(t+1) + (1 - \alpha - \beta + 2\alpha\beta) \bar{x}(t) = h(t).$$

It should be noted that (5.6) is not an autoregressive model of order 2 since $h(t)$ involves $\bar{\varepsilon}(t+1)$ at time $t+1$. To explore the properties of $\{\bar{x}_t\}_{t \in \mathbb{Z}_+}$, let $\kappa_1 = \alpha + \beta - 2$, $\kappa_2 = 1 - \alpha - \beta + 2\alpha\beta$, and the homogeneous version of (5.6) be

$$(5.7) \quad \bar{x}(t+2) + \kappa_1 \bar{x}(t+1) + \kappa_2 \bar{x}(t) = 0.$$

Then the characteristic polynomial for both (5.6) and (5.7) is

$$q(w) = w^2 + \kappa_1 w + \kappa_2,$$

which has roots $\rho_{1,2} = \frac{-\kappa_1 \pm \sqrt{\Delta_1}}{2}$ with $\Delta_1 = \kappa_1^2 - 4\kappa_2 = \Delta = \alpha^2 + \beta^2 - 6\alpha\beta$ (note that Δ is defined right before Theorem 1).

Let \mathcal{L} be the lag operator of order one, and recall $d_1 = (3 - 2\sqrt{2})\beta$ and $d_2 = (3 + 2\sqrt{2})\beta$. We have the following result that describes when $\{\bar{x}_t\}_{t \in \mathbb{Z}_+}$ can display periodic behavior and gives the solution $\{\bar{x}_t\}_{t \in \mathbb{Z}_+}$.

Theorem 4. Set $\omega = -\arctan(\kappa_1^{-1} \sqrt{|\Delta_1|})$. For model (5.7), if

$$(5.8) \quad \min\{d_1, d_2\} < \alpha < \max\{d_1, d_2\},$$

then the general, periodic solution $\{\bar{x}_t\}_{t \in \mathbb{Z}_+}$ is

$$(5.9) \quad \bar{x}(t) = c_1 |\rho_1|^t \cos(c_2 + \omega t)$$

for some constants c_1 and c_2 . If additionally

$$(5.10) \quad \frac{\beta - 1}{2\beta - 1} < \alpha < \frac{\beta}{2\beta - 1} \text{ and } \beta > \frac{1}{2}$$

or

$$(5.11) \quad \frac{\beta}{2\beta - 1} < \alpha < \frac{\beta - 1}{2\beta - 1} \text{ and } \beta < \frac{1}{2},$$

holds, then the general, periodic solution $\{\bar{x}_t\}_{t \in \mathbb{Z}_+}$ is

$$(5.12) \quad \bar{x}(t) = c_1 |\rho_1|^t \cos(c_2 + \omega t) + (1 - \rho_1 \mathcal{L})^{-1} (1 - \rho_2 \mathcal{L})^{-1} h(t),$$

where the constants c_1 and c_2 can be determined from the initial values $\bar{x}(0)$ and $\bar{x}(1)$.

Proof. WLOG, assume $d_1 = \min\{d_1, d_2\}$ and $d_2 = \max\{d_1, d_2\}$. There are three cases for the general solution to (5.7):

- (1) $\Delta_1 = 0$ if and only if $\alpha = d_1$ or $\alpha = d_2$. In this case, $\rho_1 = \rho_2 = -2^{-1}\kappa_1$ and $\bar{x}(t) = (c_0 + c_1 t) \rho_1^t$ is the general solution to (5.7) for some constants c_0 and c_1 .
- (2) $\Delta_1 > 0$ if and only if $\alpha < d_1$ or $\alpha > d_2$. In this case, $\bar{x}(t) = c_1 \rho_1^t + c_2 \rho_2^t$ is the general solution to (5.7) for some constants c_1 and c_2 .
- (3) $\Delta_1 < 0$ if and only if $d_1 < \alpha < d_2$. In this case, let $\rho_1 = |\rho_1| e^{i\omega}$ with $\omega = -\arctan\left(\kappa_1^{-1} \sqrt{|\Delta_1|}\right) \in (-\pi, \pi]$, where $i^2 = -1$. Then $\rho_1 = |\rho_1| e^{-i\omega}$ and (5.9) is the general, periodic solution to (5.7) for some constants c_1 and c_2 .

So, it is left to find a special solution to (5.6) to obtain (5.12). When $|\rho_1| < 1$, the operators $(1 - \rho_j \mathcal{L})$, $j = 1, 2$ are invertible and the inverses $(1 - \rho_j \mathcal{L})^{-1} = \sum_{s=0}^{\infty} \rho_j^s \mathcal{L}^s$ for $j = 1, 2$, where \mathcal{L}^s is the composition of \mathcal{L} by itself s times. However, $|\rho_1|^2 = \kappa_2$, and $0 < |\rho_1| < 1$ if and only if $0 < 1 - \alpha - \beta + 2\alpha\beta < 1$, which holds when $\frac{\beta-1}{2\beta-1} < \alpha < \frac{\beta}{2\beta-1}$ and $\beta > \frac{1}{2}$ or when $\frac{\beta}{2\beta-1} < \alpha < \frac{\beta-1}{2\beta-1}$ and $\beta < \frac{1}{2}$. Therefore, when additionally (5.10) or (5.11) holds, we have (5.12). This completes the proof. \square

Note that pairs (α, β) satisfying (5.8) and (5.10), or (5.8) and (5.11) in Theorem 4 do exist, which means that the solution (5.12) always exists. A trajectory from the model (5.6) is displayed in Figure 1:

A trajectory from the induced model

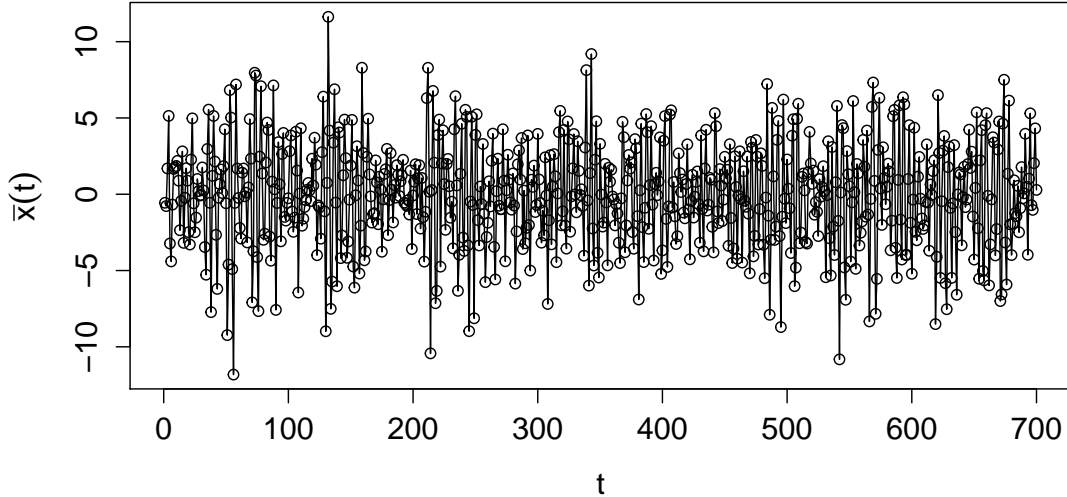


FIGURE 1. A trajectory of $\bar{x}(t)$ for $t = 0, \dots, 700$ simulated from the model (5.6) with $\alpha = 1.09804$ and $\beta = 0.7$, where $\bar{\varepsilon}(t) \propto \mathcal{N}(0, 1)$ and $\bar{\eta}(t) \propto \mathcal{N}(0, 1.6^2)$ for each t . The trajectory shows clearly that the periodicity of $\bar{x}(t)$ in (5.12) is subject to random perturbation.

6. DISCUSSION

For model (2.2), we have provided the explicit decomposition of its transition matrix \mathbf{M} , the explicit solution, and two key properties of this solution. In addition, we have provided and analyzed the solution of the model for business cycles (5.6) induced by (2.2). The explicit representations we have derived help better understand the econometric behavior to the solutions of these models and can serve as a starting point for further analysis of them.

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REFERENCES

- Ahking, F. W. and S. M. Miller (1988). Models of business cycles: A review essay. *Eastern Economic Journal* 14(2), pp. 197–202.
- Burns, A. F. and W. C. Mitchell (1946). *Measuring Business Cycles*. Cambridge, MA, USA: National Bureau of Economic Research.
- Jacobson, N. (1953). *Lectures in Abstract Algebra*, Volume II, Linear Algebra. Springer Science+Business Media, LLC.
- Kydland, F. E. and E. C. Prescott (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy* 85(3), pp. 473–492.
- Kydland, F. E. and E. C. Prescott (1982). Time to build and aggregate fluctuations. *Econometrica* 50(6), pp. 1345–1370.
- Lucas, R. E. (1977). Understanding business cycles. *Carnegie-Rochester Conference Series on Public Policy* 5(1), 7–29.
- Lucas, R. E. (1991). *Models of Business Cycles*. Wiley-Blackwell.
- Ormerod, P. (1998). *Butterfly Economics: A New General Theory of Social and Economic Behavior*. London, UK: Faber and Faber Ltd.
- Ormerod, P. (2001). The Keynesian micro-foundations of the business cycle: Some implications of globalization. In P. Arestis, M. Baddeley, and J. McCombie (Eds.), *What Global Economic Crisis?*, pp. 203–218. London, UK: Macmillan Publishers Ltd.

CENTER FOR STATISTICS AND MACHINE LEARNING AND LEWIS-SIGLER INSTITUTE FOR INTEGRATIVE GENOMICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA

E-mail address: xiongzhi@princeton.edu